

# Inflation from a massive scalar field and scalar perturbations of the metric

F. Lara, F. Astorga, M. Bellini

*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,  
AP:2-82, (58041) Morelia, Michoacán, México.*

## Abstract

In the framework of chaotic inflation we study the case of a massive scalar field using a semiclassical approach. We derive the energy density fluctuations in the infrared sector generated by matter field and gauge-invariant metric fluctuations by means of a different method. We find that the super Hubble density fluctuations increase during inflation for a massive scalar field, in agreement with the result obtained in a previous work in which a power-law expanding universe was considered.

## I. INTRODUCTION

Inflationary cosmology was developed by Guth in the 80's [1] in order to solve some of the shortcomings of the big bang theory, and in particular to explain the extraordinary homogeneity of the observable universe. However, the universe after inflation in this scenario becomes very inhomogeneous. Following a detailed investigation of this problem, A. Guth and E. Weinberg concluded that the old inflation model could not be improved [2].

These problems were sorted out by A. Linde in 1983 with the introduction of chaotic inflation [3]. In this scenario inflation can occur in theories with potentials such as  $V(\varphi) \sim \varphi^n$ . It may begin in the absence of thermal equilibrium in the early universe, and it may start even at the Planckian density, in which case the problem of initial conditions for inflation can be easily solved [4]. According to the simplest versions of chaotic inflationary theory the universe is not a single expanding ball of fire produced in the big bang, but rather a huge eternally growing fractal. It consists of many inflating balls that produce new balls, which produce more new balls, ad infinitum.

Within this context, the Cosmic Microwave Background Radiation plays a fundamental role. Data coming from COBE and the other CMB radiation detectors have started shifting inflation from a mainly theoretical subject to a more constrained field. It is expected that new observations from the MAP, to be launched this summer, and Planck missions, planned for 2007, are crucial for the understanding of an inflationary stage in the universe. With the advent of new CMBR physics, it is thought that the measurements of the power spectrum of temperature fluctuations at much smaller angular separations, of the order of fractions of a degree, might be sensitive to the standard model parameters with unprecedented precision. This will bring a remarkable change in the field.

The dynamics of a scalar field  $\varphi$  minimally coupled to a classical gravitational one is described by the Lagrangian:

$$\mathcal{L}(\varphi, \varphi_{,\mu}) = -\sqrt{-g} \left[ \frac{R}{16\pi} + \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + V(\varphi) \right] . \quad (1)$$

If the spacetime has a Friedman-Robertson-Walker (FRW) metric,  $ds^2 = -dt^2 + a^2(t)dr^2$ , which describes a globally isotropic and homogeneous universe, the equations of motion for the field operator  $\varphi$  and the Hubble parameter  $H \equiv \frac{\dot{a}}{a}$  are:

$$\ddot{\varphi} - \frac{1}{a^2} \nabla^2 \varphi + 3H\dot{\varphi} + V'(\varphi) = 0 , \quad (2)$$

$$H^2 = \frac{4\pi}{3M_p^2} \left\langle \dot{\varphi}^2 + \frac{1}{a^2} (\vec{\nabla} \varphi)^2 + 2V(\varphi) \right\rangle , \quad (3)$$

where the overdot denotes the time derivative and  $V'(\varphi) \equiv \frac{dV}{d\varphi}$ . The expectation value is assumed to be a constant function of the spatial variables for consistency with the FRW metric. When inflation ends the field starts oscillating rapidly and its potential energy is converted into thermal energy. This is the general scheme of the inflationary scenario without considering the quantum effects.

In this work we consider a semiclassical expansion of the theory. To obtain this we decompose the inflaton operator in a classical field  $\phi_c$  plus the quantum fluctuations  $\phi$ ,  $\varphi = \phi_c + \phi$ . The spatially homogeneous field  $\phi_c(t)$  is defined as the solution to the classical equation of motion [here,  $H(\varphi) \equiv H_c(\phi_c)$ ]:

$$\ddot{\phi}_c + 3H_c \dot{\phi}_c + V'(\phi_c) = 0, \quad (4)$$

where the prime denotes the derivative with respect to  $\phi_c$ . The evolution of the quantum operator  $\phi$  is given by:

$$\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + 3H_c(\phi_c) \dot{\phi} + V''(\phi_c) \phi = 0, \quad (5)$$

where  $H_c(\phi_c) = \frac{\dot{a}}{a}$  is the Hubble parameter and the classical Friedmann equation is

$$H_c^2 = \frac{4\pi}{3M_p^2} \left[ \dot{\phi}_c^2 + 2V(\phi_c) \right] . \quad (6)$$

From eqs.(4) and (6), we obtain the classical dynamics of the Hubble parameter and the inflaton field given by the following relations:

$$\dot{\phi}_c = -\frac{M_p^2}{4\pi} H'_c , \quad (7)$$

$$\dot{H}_c = H'_c \dot{\phi}_c = -\frac{M_p^2}{4\pi} (H'_c)^2 . \quad (8)$$

Hence, from eq. (3) we derive the expression for the scalar potential  $V(\phi_c)$

$$V(\phi_c) = \frac{3M_p^2}{8\pi} \left( H_c^2 - \frac{M_p^2}{12\pi} (H'_c)^2 \right) . \quad (9)$$

Equations (7) and (8) define the classical evolution of space-time, and determine the relation between the classical potential and the inflationary regimes. On the other hand eq.(5) defines the quantum dynamics of the field  $\phi$ .

If during inflation the slow-roll condition is imposed, the following conditions are required [6]

$$\gamma = \frac{2}{\kappa^2} \left( \frac{H'_c}{H_c} \right)^2, \quad (10)$$

$$\eta = \frac{2}{\kappa^2} \frac{H''_c}{H_c}, \quad (11)$$

where  $\kappa^2 = \frac{8\pi}{M_p^2}$  and  $M_p \simeq 1.210^{19} \text{ GeV}$  is the Planckian mass. Furthermore, to solve the flatness problem 60 or more e-folds are required. The number of e-folds during inflation is

$$N = \frac{a(t_f)}{a(t_i)} = \int_{t_i}^{t_f} H_c(t) dt, \quad (12)$$

where  $a$  is the scale factor of the universe and  $t_i$  and  $t_f$  are respectively the times when inflation starts and ends.

In this work we consider an inflaton field with mass  $m = 10^{-5} M_p$ .

## II. GAUGE-INVARIANT METRIC FLUCTUATIONS

The issue of gauge-invariance becomes critical when we attempt to analyze how the scalar metric perturbations produced in the very early universe influence a globally flat FRW background metric. For a diagonal stress  $T_{ij}$  the perturbed FRW metric is described by [5]

$$ds^2 = (1 + 2\psi) dt^2 - a^2(t) (1 - 2\psi) dx^2, \quad (13)$$

where  $\psi$  are the perturbations of the metric.

After linearizing the Einstein equations in terms of  $\phi$  and  $\psi$ , one obtains

$$\ddot{\psi} + \left( \frac{\dot{a}}{a} - 2 \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right) \dot{\psi} - \frac{1}{a^2} \nabla^2 \psi + 2 \left[ \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right] \psi = 0, \quad (14)$$

$$\frac{1}{a} \frac{d}{dt} (a\psi)_{,\beta} = \frac{4\pi}{M_p^2} (\dot{\phi}_c \phi)_{,\beta}, \quad (15)$$

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V''(\phi_c) \phi + 2V'(\phi_c) \psi - 4\dot{\phi}_c \dot{\psi} = 0, \quad (16)$$

where the dynamics of  $\phi_c$  is given by the eqs. (4) and (7). Eq. (14) can be simplified by introducing the field  $h = e^{1/2 \int [\dot{a}/a - 2\ddot{\phi}_c/\dot{\phi}_c] dt} \psi$

$$\begin{aligned} \ddot{h} - \frac{1}{a^2} \nabla^2 h - \left[ \frac{1}{4} \left( \frac{\dot{a}}{a} - 2 \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right)^2 + \frac{1}{2} \left( \frac{\ddot{a}a - \dot{a}^2}{a^2} - \frac{2 \frac{d}{dt} (\ddot{\phi}_c \dot{\phi}_c) - 4 \dot{\phi}_c^2}{\dot{\phi}_c^2} \right) \right. \\ \left. - 2 \left( \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right) \right] h = 0. \end{aligned} \quad (17)$$

This field can be expanded in terms of the modes  $h_k = e^{i\vec{k}\cdot\vec{x}}\tilde{\xi}_k(t)$

$$h(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[ \alpha_k h_k + \alpha_k^\dagger h_k^* \right], \quad (18)$$

where  $\alpha_k$  and  $\alpha_k^\dagger$  are the annihilation and creation operators with commutation relations

$$[\alpha_k, \alpha_{k'}^\dagger] = \delta^{(3)}(k - k'), \quad (19)$$

$$[\alpha_k, \alpha_{k'}] = [\alpha_k^\dagger, \alpha_{k'}^\dagger] = 0. \quad (20)$$

The equation for the modes  $\tilde{\xi}_k$  is

$$\ddot{\tilde{\xi}}_k + \tilde{\omega}_k^2(t) \tilde{\xi}_k = 0, \quad (21)$$

where  $\tilde{\omega}_k^2 = k^2/a^2 - \tilde{k}_0^2/a^2$  is the squared time dependent frequency and  $\tilde{k}_0$  separates the infrared and ultraviolet sectors, and is given by

$$\frac{\tilde{k}_0^2}{a^2} = \frac{1}{4} \left( \frac{\dot{a}}{a} - 2 \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right)^2 + \frac{1}{2} \left( \frac{\ddot{a}a - \dot{a}^2}{a^2} - \frac{2 \frac{d}{dt} (\ddot{\phi}_c \dot{\phi}_c) - 4 \ddot{\phi}_c^2}{\dot{\phi}_c^2} \right) - 2 \left[ \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \frac{\ddot{\phi}_c}{\dot{\phi}_c} \right]. \quad (22)$$

Note that for super Hubble scales ( $k^2 \ll \tilde{k}_0^2$ ), eq. (21) can be approximated by

$$\ddot{\tilde{\xi}}_0 - \frac{\tilde{k}_0^2}{a^2} \tilde{\xi}_0 = 0, \quad (23)$$

where  $\tilde{\xi}_0$  denotes the zero-mode  $\tilde{\xi}_{k=0}(t)$ .

Finally, the fluctuations for the energy density are [8]

$$\frac{\delta\rho}{\rho} = -2\psi. \quad (24)$$

### III. THE MASSIVE SCALAR FIELD

We consider a massive scalar field with potential

$$V(\phi_c) = \frac{m^2}{2} \phi_c^2. \quad (25)$$

If the slow rolling conditions (10) and (11) are fulfilled, the second term inside the brackets of eq. (9) can be neglected, so that the Hubble parameter is given by

$$H_c(\phi_c) = 2\sqrt{\frac{\pi}{3}} \frac{m}{M_p} \phi_c. \quad (26)$$

From eq. (7) one obtains the time evolution for the scalar field

$$\phi_c(t) = \phi_i - \frac{mM_p}{2\sqrt{3}\pi}t, \quad (27)$$

where  $\phi_i \equiv \phi_c(t_i)$  is the value of the homogeneous field when inflation starts. The scale factor, written as a function of  $\phi_c$ , is expressed as

$$a(\phi_c) = a_i e^{-\frac{2\pi}{M_p^2}\phi_c^2}, \quad (28)$$

and increases with time due to the fact that  $\phi_c$  decreases with time. Here,  $a_i \equiv a(t_i)$ . From eqs. (26), (10) and (11) one obtains the slow roll parameters

$$\gamma = \frac{M_p^2}{4\pi} \phi_c^{-2}, \quad (29)$$

$$\eta = 0, \quad (30)$$

which complies with the required conditions for slow-rolling for  $\phi_c \gg \frac{M_p}{2\sqrt{\pi}}$ .

### A. Super Hubble matter field fluctuations

The equation of motion (5) for the matter field fluctuations  $\phi$  can be simplified by means of the change of variable  $\chi = a^{3/2}\phi$ . With this transformation the following equation for the redefined field  $\chi$  becomes

$$\ddot{\chi} + \left[ \frac{k^2}{a^2} - \frac{k_0^2}{a^2} \right] \chi = 0, \quad (31)$$

where  $k_0$  is the wavenumber that separates the infrared ( $k^2 \ll k_0^2$ ) and ultraviolet ( $k^2 \gg k_0^2$ ) sectors. The squared time dependent parameter of mass  $\mu^2(t) = \frac{k_0^2}{a^2}$  is given by

$$\mu^2(t) = \frac{9}{4}H_c^2 + \frac{3}{2}\dot{H}_c - V_c''. \quad (32)$$

The redefined matter field fluctuations  $\chi(\vec{x}, t)$  for the scalar field in the infrared sector can be written as a Fourier expansion in terms of the modes  $\chi_k = e^{i\vec{k} \cdot \vec{x}} \xi_k(t)$

$$\chi_{cg}(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \theta(k - \epsilon k_0) \left[ a_k \chi_k + a_k^\dagger \chi_k^* \right], \quad (33)$$

where  $a_k$  and  $a_k^\dagger$  are the annihilation and creation operators, which satisfy the commutation relations [7]:  $[a_k, a_{k'}^\dagger] = \delta^{(3)}(k - k')$ ,  $[a_k^\dagger, a_{k'}^\dagger] = [a_k, a_{k'}] = 0$ . Furthermore,  $\epsilon \ll 1$  is a dimensionless constant [7], which is of the order  $\epsilon \simeq 10^{-3} - 10^{-4}$ .

From eq. (27), one can write the equation of motion for the time dependent modes  $\xi_k$  as a function of the homogenous scalar field  $\phi_c$

$$\xi_k''(\phi_c) + \left[ \frac{12\pi k^2}{(mM_p)^2 a^2} - 4M^2 \phi_c^2 + 6M \right] \xi_k(\phi_c) = 0, \quad (34)$$

where the prime denotes the derivative with respect to  $\phi_c$  and  $M = \frac{3\pi}{M_p^2}$ . Since we are interested in studying the super Hubble ( $k^2 \ll k_0^2$ ) matter field fluctuations, the equation (34) takes the form

$$\xi_0'' - [4M^2\phi_c^2 - 6M]\xi_0 = 0. \quad (35)$$

The solutions of this equation describe the infrared sector and thus must be real [9] to obtain classicality conditions of matter field fluctuations in this sector. Hence, the real solution for  $\xi_k$  comes out to be

$$\xi_0(\phi_c) = |c_1| e^{-M\phi_c^2} \phi_c, \quad (36)$$

where  $c_1$  is an arbitrary constant. The squared amplitude for the infrared matter field fluctuations  $\phi_{cg} = a^{-3/2}\chi_{cg}$  are  $\langle\phi_{cg}^2\rangle_{IR} = \frac{a^{-3}}{2\pi^2} \int_0^{\epsilon k_0} dk k^2 \xi_0^2$ , so that

$$\langle\phi_{cg}^2\rangle_{IR} \simeq \frac{c_1^2 \epsilon^3}{6\pi^2} [4M^2\phi_c^2 - 6M]^{3/2} \phi_c^2 e^{-2M\phi_c^2}. \quad (37)$$

For inflation to take place one requires that  $4M^2\phi_c^2 - 6M > 0$ , so that  $\phi_c^2 > \frac{M^2}{2\pi}$ . Furthermore, the energy density fluctuations  $\frac{\delta\rho}{\rho} \simeq \frac{V'_c}{V_c} \langle\phi_{cg}^2\rangle^{1/2}$  are

$$\frac{\delta\rho}{\rho} \simeq \frac{c_1 \epsilon^{3/2}}{\sqrt{6}\pi} [4M^2\phi_c^2 - 6M]^{3/4} e^{-M\phi_c^2}, \quad (38)$$

which agree with the COBE data,  $\frac{\delta\rho}{\rho} \simeq 10^{-5}$ , for  $c_1 \simeq 208.71$ . The power spectrum is given by  $\mathcal{P}_{\phi_{cg}} = |\delta_k|^2$ , for  $\langle\phi_{cg}^2\rangle_{IR} = \int_0^{\epsilon k_0} \frac{dk}{k} \mathcal{P}_{\phi_{cg}}$ , such that the spectral density  $\delta_k$  for this model is

$$\delta_k = \frac{c_1 \epsilon^{3/2}}{\sqrt{6}\pi} [4M^2\phi_c^2 - 6M] e^{-M\phi_c^2} k^{3/2}. \quad (39)$$

The spectrum  $n - 1 = -6\gamma + 2\eta$  is given by [see eqs. (10) and (11)]

$$|n - 1| = \frac{3M_p^2}{2\pi} \phi_c^{-2}, \quad (40)$$

which is very close to the Harrison - Zeldovich spectrum (i.e.,  $n = 1$ ) [10] for  $\phi_c \gg M_p$  and is in good agreement with CMB data spectrum  $|n - 1.2| < 0.3$  [11].

## B. Super Hubble metric fluctuations for a massive scalar field

Since  $\phi_c(t) = \phi_i - \frac{mM_p}{2\sqrt{3}\pi}t$ , the eq. (23) for  $\tilde{\xi}_0(\phi_c)$  can be written as

$$\tilde{\xi}_0'' - (4L^2\phi_c^2 + 6L)\tilde{\xi}_0 = 0, \quad (41)$$

where  $L = \frac{\pi}{M_p^2}$ . This has the real solution

$$\tilde{\xi}_0(\phi_c) = e^{L\phi_c^2} \left[ D_1 \phi_c + D_2 e^{-2L\phi_c^2} + D_2 \sqrt{2\pi L} \phi_c \operatorname{Erf}[\sqrt{2L} \phi_c] \right], \quad (42)$$

where  $(D_1, D_2)$  are real constants and  $\operatorname{Erf}$  is the error function. Hence, the amplitude of fluctuations for the metric perturbations for super Hubble scales is

$$\langle \psi_{cg}^2 \rangle \simeq \frac{a^{-1}}{2\pi^2} \int_0^{\epsilon k_0} dk \, k^2 \left[ \tilde{\xi}_0(\phi_c) \right]^2, \quad (43)$$

and the fluctuations for energy density are [see eq. (24)]

$$\begin{aligned} \left. \frac{\delta\rho}{\rho} \right|_{IR} &\simeq 2 \langle \psi_{cg}^2 \rangle^{1/2} \simeq \frac{2a\epsilon^{3/2}}{\sqrt{6\pi}} \tilde{\xi}_0(\phi_c) \mu^{3/2} \\ &= \frac{2a_i e^{-L\phi_c^2}}{\sqrt{6\pi}} \epsilon^{3/2} \left( 4L^2 \phi_c^2 + 6L \right)^{3/4} \left[ D_1 \phi_c + D_2 \left( e^{-2L\phi_c^2} + \sqrt{2\pi L} \phi_c \operatorname{Erf}[\sqrt{2L} \phi_c] \right) \right], \end{aligned} \quad (44)$$

which increase when  $\phi_c$  decreases (i.e., when the time increases). This means that super Hubble scalar metric fluctuations increase during inflation for a massive scalar field.

#### IV. FINAL COMMENTS

In this work we studied the evolution of both, super Hubble matter and metric fluctuations during inflation. We have considered the case of a massive scalar field with a potential given by  $V(\phi_c) = \frac{m^2}{2} \phi_c^2$ . The fact that the modes for the redefined fields  $\chi_{cg}$  and  $h_{cg}$  can be mapped as functions of  $\phi_c$ , makes it possible to find solutions for the zero modes ( $k = 0$ ) for  $\xi_k$  and  $\tilde{\xi}_k$ , and thus  $\langle \phi_{cg}^2 \rangle$  and  $\langle \psi_{cg}^2 \rangle$  can be calculated by means of a semiclassical method.

We find that super Hubble matter field fluctuations increase during inflation so that at the end of inflation  $\delta\rho/\rho \simeq 10^{-5}$  is reached. Furthermore, we get a spectral index  $n \simeq 1$ , which describes a nearly scale invariant spectrum for super Hubble matter field fluctuations. Finally, we find that scalar metric fluctuations increase during inflation. This behaviour agrees with results from a previous work in which the IR scalar metric perturbations are studied for a power-law expanding universe [12]. It is interesting to point out, however, that metric perturbations are not amplified after  $m^2 \phi^2$  inflation. As it has been recently shown [13], they cease to grow during reheating.

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